Magnetic Monopole in the Loop Representation

Lorenzo Leal[a][*] and Alexander López [b, a][†]

a Grupo de Campos y Partículas, Departamento de Física, Facultad de Ciencias, Universidad Central de Venezuela, AP 47270, Caracas 1041-A, Venezuela

b Centro de Física, Instituto Venezolano de Investigaciones Científicas, AP 21827, Caracas 1020-A, Venezuela

Abstract

We quantize, within the Loop Representation formalism, the electromagnetic field in the presence of a static magnetic pole. It is found that loop-dependent physical wave functionals acquire a topological dependence on the surfaces bounded by the loop. This fact generalizes what occurs in ordinary quantum mechanics in multiply connected spaces. When Dirac's quantization condition is satisfied, the dependence on the surfaces disappears, together with the influence of the monopole on the quantized electromagnetic field.

I. INTRODUCTION

Dirac [1,2] found that the mere existence of a single monopole would explain the quantized nature of the electric charge. He discovered a relation between the unit of electric charge and that of the magnetic pole, which is currently known as Dirac's quantization condition. In rationalized units ($c = \hbar = 1$) it reads

$$\frac{eg}{4\pi} = \frac{1}{2}n, \tag{1}$$

where n is an integer and e (g) is the unit of electric (magnetic) charge. Besides this remarkable prediction, the Dirac theory of magnetic monopoles has been a source of inspiration for the development of new ideas in theoretical physics. Often, it happens that different approaches to understand the formulation of Dirac bring out novelties or unexpected relationships between old things.

Being a gauge theory, the Dirac theory of magnetic poles should be a candidate to admit a quantum geometric representation, such as the Loop Representation (LR) of Maxwell theory [3–5]. In this article we address this point to some extent. Concretely, we study the LR

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[*]lleal@fisica.ciens.ucv.ve

[†]alopez@ivic.ve

formulation of quantum Maxwell theory in presence of an external magnetic pole, taking as starting point a first order action due to Schwinger [6], which is based on the earlier Dirac theory [1,2]. We shall see that the LR formulation of the Maxwell theory with a static monopole corresponds to that of the *free theory*, except by the fact that the loop-dependent wave functional acquires a topological dependence on the manner that the loop "winds" around the monopole. This dependence is manifested through a topological phase factor picked up by the wave functional when the loop undergoes an adiabatic excursion in the presence of the monopole. It could be said that loop-dependent wave functionals become multivalued in the presence of the monopole, in the same sense that in ordinary quantum mechanics wave functions are allowed to be multivalued whenever the configuration space is multiply connected [7–9]. This results are also related with previous studies about the quantum theory of strings in presence of a Kalb-Ramond vortex [10,11] where a generalization of the concept of anyons can be envisaged.

In the next section we present the model and discuss its quantization in the LR formulation. In the last one we study in which sense the magnetic monopole turns Maxwell theory into a loop-dependent theory with non-trivial boundary conditions in loop-space.

II. QUANTIZATION AND LOOP REPRESENTATION

Electromagnetism with magnetic charges can be studied from the first order Schwinger action [6]

$$S = \int dx^4 \left(A_\mu J_e^\mu + B_\mu J_m^\mu - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right), \tag{2}$$

where B is given by

$$B_\mu(x) = \int dy^4 * F_{\mu\nu}(y) f^\nu(y - x) + \partial_\mu \lambda(x). \tag{3}$$

Here, f obeys

$$\partial_\mu f^\mu(y) = \delta^4(y), \tag{4}$$

and λ is an arbitrary function. The dual $*F$ is given by

$$*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \tag{5}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetric symbol and J_e (J_m) denote the electric (magnetic) current density. The independent fields in (2) are A_ν and $F_{\mu\nu}$. Varying (2) with respect to A_μ gives

$$\partial_\nu F^{\mu\nu} = J_e^\mu, \tag{6}$$

whereas variations with respect to $F_{\mu\nu}$ yield

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \int dy^4 J_m^\alpha(y) f^\beta(x - y), \tag{7}$$

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which, with the use of equations (3) and (4), implies

$$\partial_\nu * F^{\mu\nu} = J_m^\mu. \tag{8}$$

Thus, one obtains the Maxwell equations with both electric and magnetic currents. The duality electricity-magnetism manifests through the invariance of the equations under the rotations

$$J_e \rightarrow \cos\phi J_e + \sin\phi J_m, \tag{9}$$

$$J_m \rightarrow -\sin\phi J_e + \cos\phi J_m, \tag{10}$$

$$F \rightarrow \cos\phi F + \sin\phi * F. \tag{11}$$

We are interested in studying how the presence of a magnetic monopole affects the loop-space formulation of the Maxwell field. Hence, we take $J_e = 0$ in equation (2) and restrict ourselves to consider a static monopole (we take it at the origin of space), which forces the magnetic current to be written as

$$J_m^\mu(x) = g\delta_0^\mu \delta^3(\vec{x}). \tag{12}$$

A convenient choice for f_μ (fulfilling equation (4)) is

$$f^\nu(y) = -\frac{1}{4\pi}\frac{y^i}{\vec{y}^3}\delta_i^\nu \delta(y_0). \tag{13}$$

The Hamiltonian formulation begins with the definition of the canonical momenta associated to A_0, F_{ij} and A_i, which result to be

$$\Pi^0(x) \approx 0, \tag{14}$$

$$\Pi^{ij}(x) \approx 0, \tag{15}$$

$$\Pi^i(x) = -F^{0i}(x), \tag{16}$$

respectively. Equations (14), (15) are (primary) constraints in the sense of Dirac [12], and we have introduced the weak equality symbol $\approx$ to mean that these equalities should not be used until Poisson brackets are calculated. Since F^{0i} is already a conjugate momentum, it is not necessary to treat it as a coordinate and to define its own canonical momentum [13]. Following Dirac's method to deal with constrained systems one constructs the total Hamiltonian [12]

$$H^* = H + \int d\vec{x}^3\, u(x)\Pi^0(x) + \int d\vec{x}^3\, u_{ij}(x)\Pi^{ij}(x), \tag{17}$$

where $u(x)$ and $u_{ij}(x)$ are Lagrange multipliers and H is the canonical Hamiltonian

$$H = \int d\vec{x}^3 \left[\frac{1}{2}\Pi_i^2 - \frac{1}{4}F_{ij}^2 + \frac{1}{2}F_{ij}\left(f_{ij} - b_{ij}\right)\right] + \int d\vec{x}^3 \partial_i \Pi^i A_0. \tag{18}$$

Here we have defined

$$f_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x), \tag{19}$$

$$b_{ij}(x) = g\epsilon_{ijk} f^k(\vec{x}). \tag{20}$$

It is worth noticing that when the static monopole is absent this Hamiltonian properly reduces to that of the conventional Maxwell theory, since in that case the magnetic field F_{ij} and the curl of the potential f_{ij} coincide.

The non-vanishing equal time canonical Poisson brackets are given by

$$\{F_{ij}(x), \Pi^{kl}(y)\} = \frac{1}{2}\left(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k\right)\delta^3(\vec{x}-\vec{y}). \tag{21}$$

$$\{A_\alpha(x), \Pi^\beta(y)\} = \delta_\beta^\alpha \delta^3(\vec{x}-\vec{y}). \tag{22}$$

Following Dirac [12], one must impose the time preservation of the constraints. From (14) one finds, as a secondary constraint, the Gauss Law

$$\partial_i \Pi^i(x) \approx 0. \tag{23}$$

This constraint, in turn, has vanishing Poisson bracket with the total Hamiltonian, hence, it does not produce further constraints. On the other hand, time preservation of (15) gives the secondary constraint

$$K_{ij}(x) = F_{ij}(x) - f_{ij}(x) + b_{ij}(x) \approx 0. \tag{24}$$

Finally, the preservation of constraint (24) allows to obtain the Lagrange multipliers u_{ij} in terms of the momenta Π^i

$$u_{ij}(x) = -\frac{1}{2}\left(\partial_i F_{0j}(x) - \partial_j F_{0i}(x)\right), \tag{25}$$

which must be substituted into the expression for the total Hamiltonian.

It is found that constraints (23) and (14) are first class, while (15) and (24) are second class. Following Dirac's procedure we must introduce Dirac brackets in order to obtain a quantum theory consistent with these second class constraints. Though it is not difficult to carry out the calculations of the matrix of the Poisson brackets between second class constraints and its inverse, which are the ingredients needed for building Dirac brackets, a simple argument suffices to get the result. Since Dirac brackets are going to be consistent with second class constraints, these may be put as strong equalities. Hence, we can write F_{ij} and its momentum Π^{ij} in terms of the remaining canonical variables, and substitute these expressions in the total Hamiltonian. Once this is done, we have only to consider Dirac brackets between the canonical variables that remain, which are A_i, A_0, and their canonical conjugates. But it is easy to see that these Dirac brackets just coincide with the Poisson ones. The net result is that we can eliminate F_{ij} and Π^{ij} using equations (15) and (24), and continue using the Poisson brackets for the remaining variables.

At this point, it is also convenient to "eliminate" the constraint (14) of the formalism. This can be accomplished by fixing the temporal gauge $A_0 = 0$, and treating this equation as a new constraint, which, together with (14), can be considered as a pair of second class constraints. Then, we can put A_0 and Π^0 as strongly vanishing. As before, it can be seen

that the new Dirac brackets are equal to the Poisson ones, as far as we consider only the remaining variables, namely A_i and their canonical conjugates.

Now we are ready to quantize the theory. First, promote canonical variables to operators obeying equal time canonical commutators

$$[\hat{A}_i(x), \hat{A}_j(y)] = 0, \tag{26}$$

$$\left[\hat{\Pi}^i(x), \hat{\Pi}^j(y)\right] = 0, \tag{27}$$

$$\left[\hat{A}_i(x), \hat{\Pi}^j(y)\right] = i\delta_i^j \delta^3(\vec{x} - \vec{y}). \tag{28}$$

The first class constraints define the physical states $|\Psi >$ as those that satisfy

$$\partial_i \hat{\Pi}^i(x)|\Psi > = 0. \tag{29}$$

On the physical subspace, the dynamics is given by the Schrödinger equation

$$i\partial_t |\Psi_t > = \hat{H}|\Psi_t >, \tag{30}$$

with

$$\hat{H} = \int d\vec{x}^3 \left[\frac{1}{2}\hat{\Pi}_i^2 + \frac{1}{4}\left(\partial_i \hat{A}_j - \partial_j \hat{A}_i - b_{ij}\right)^2\right]. \tag{31}$$

Thus, we obtain that the static monopole manifests in the theory just through the external field b_{ij}, that must be subtracted from the curl of the vector potential to give the magnetic field operator.

We are now prepared to discuss the LR of the model. We begin by recalling that the Abelian path space (PS) can be defined as the set of certain equivalence classes of curves γ in (for our purposes) R^3 [3–5,14]. The equivalence relation is given by the so called *form factor* $T^i(\vec{x}, \gamma)$ of the curve

$$T^i(\vec{x}, \gamma) = \int_\gamma dy^i \delta(\vec{x} - \vec{y}), \tag{32}$$

as follows: γ and γ, are said to be equivalent (i.e., represent the same path) if their form factors coincide. Closed curves give raise to a subspace of the PS: the loop space. It can be seen that the usual composition of curves translates into a composition of paths that endows the PS with a group structure.

The path representation arises when one considers path dependent wave functionals $\Psi[\gamma]$, and realizes the canonical field operators by means of operations onto these wave functionals [3–5,14]. We define the path and loop derivatives $\delta_i(\vec{x})$ and $\Delta_{ij}(\vec{x})$ by

$$\left(1 + u^i(\vec{x})\delta_i(\vec{x})\right)\Psi[\gamma] = \Psi[\gamma \circ u], \tag{33}$$

$$\left(1 + \frac{1}{2}\sigma^{ij}(\vec{x})\Delta_{ij}(\vec{x})\right)\Psi[\gamma] = \Psi[\gamma \circ \delta c], \tag{34}$$

where $\circ$ denotes the PS product [4]. The derivative $\delta_i(\vec{x})$ $(\Delta_{ij}(\vec{x}))$ measures the change in the path-dependent wave functional when an infinitesimal path δu (infinitesimal loop δc) is

attached to its argument γ at the point $\vec{x}$. It is understood that these changes are considered up to first order in the infinitesimal vector u^i associated with the small path, or with the surface element

$$\sigma^{ij} = u^i v^j - v^j u^i, \tag{35}$$

generated by the infinitesimal vectors $\vec{u}$ and $\vec{v}$ that define the small loop δc. It can be shown that both derivatives are related by [3–5,14]

$$\partial_i \delta_j(\vec{x}) - \partial_j \delta_i(\vec{x}) = \Delta_{ij}(\vec{x}). \tag{36}$$

With these tools at hand we represent the canonical field as operators acting on path dependent wave functionals $\Psi[\gamma]$ by means of the prescriptions

$$\hat{\Pi}^i(\vec{x}) \rightarrow e T^i(\vec{x}, \gamma), \tag{37}$$

$$\hat{A}_j(\vec{x}) \rightarrow \frac{i}{e} \delta_j(\vec{x}). \tag{38}$$

It is readily seen that this realizes the algebra (26)- (28). We see that in this representation the form factor corresponds to Faraday lines of electric field. The magnetic field operator, in turn, appends a small closed line of electric field to the argument of the wave functionals. The constant e is introduced to fix the scale of the Faradays lines of electric field. In four dimensions (and using natural units) this constant is dimensionless (as well as the magnetic charge g) and can be taken as the elementary electric charge. Since the divergence of the form factor $T^i(\vec{x}, \gamma)$ vanishes when the path is closed, the Gauss constraint (29) is identically satisfied if we restrict to loop-dependent wave functionals [3]. The Hamiltonian in the LR is then given by

$$H = \int d\vec{x} \left[\frac{1}{2} e^2 \left(T^i(\vec{x}, \gamma) \right)^2 - \frac{1}{4e^2} \left(i\Delta_{ij}(\vec{x}) - e\, b_{ij}(\vec{x}) \right)^2 \right]. \tag{39}$$

When $g = 0$, H reduces to the Hamiltonian of free electromagnetism in the LR [3–5], as it should be.

III. MULTIVALUED LOOP-DEPENDENT WAVE FUNCTIONALS

We have seen that introducing a static monopole in quantum Maxwell theory, in the LR, amounts to replacing the loop derivative $\Delta_{ij}(\vec{x})$ by a kind of "covariant derivative"

$$i\Delta_{ij}(\vec{x}) \rightarrow i\Delta_{ij}(\vec{x}) - e\, b_{ij}(\vec{x}). \tag{40}$$

Now, we shall see that it is possible to recast the Schrödinger equation as that corresponding to a *free* theory, provided that we deal with *multivalued loop-dependent* wave functionals. To see how this happens we find it convenient to employ the space of surfaces framework [15], which we summarize following very closely reference [16]. One starts with the space of piecewise smooth oriented surfaces Σ in R^3. We define two surfaces as equivalent if they share the same "surface form-factor"

$$T^{ij}(\vec{x}, \Sigma) = \int d\Sigma_y^{ij}\, \delta^{(3)}(\vec{x} - \vec{y}). \tag{41}$$

Here $d\Sigma_y^{ij}$ is the surface element

$$d\Sigma_y^{ij} = \left(\frac{\partial y^i}{\partial s}\frac{\partial y^j}{\partial r} - \frac{\partial y^i}{\partial r}\frac{\partial y^j}{\partial s}\right)dsdr, \tag{42}$$

with s, r being surface parameters. Now we consider functionals $\Psi[\Sigma]$ and introduce the surface derivative $\delta_{ij}(\vec{x})$, that measures the response of $\Psi[\Sigma]$ when an element of surface whose infinitesimal area is σ_{ij} is attached to the argument Σ of $\Psi[\Sigma]$ at the point x, up to first order in σ_{ij} :

$$\Psi[\delta\Sigma \cdot \Sigma] = (1 + \sigma^{ij}\delta_{ij}(\vec{x}))\Psi[\Sigma]. \tag{43}$$

The surface derivative $\delta_{ij}(\vec{x})$ and the loop derivative $\Delta_{ij}(\vec{x})$ are different things. However, since in R^3 loop-dependence is a particular case of surface-dependence (a loop is always the boundary of an open surface in R^3), the loop derivative can be seen as the surface derivative restricted to loop-dependent functionals. Hence it makes sense to surface-derive loop-dependent quantities. Soon we shall make use of the surface-derivative of the form factor

$$\delta_{ij}(\vec{x})T^{kl}(\vec{y}, \Sigma) = \frac{1}{2}\left(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k\right)\delta^{(3)}(\vec{x} - \vec{y}). \tag{44}$$

Turning back to our model, let us consider an open surface Σ whose boundary coincides with γ. Then define, from the path-dependent wave functional $\Psi[\gamma]$, the surface-dependent one

$$\Psi[\Sigma] \equiv \exp\left(ie\int d\Sigma_{\vec{y}}^{km}b_{km}(\vec{y})\right)\Psi[\gamma] \tag{45}$$

$$= \exp(\frac{ieg}{4\pi}\Omega(\Sigma))\,\Psi[\gamma], \tag{46}$$

where $\Omega(\Sigma)$ is the solid angle subtended by Σ, measured from the monopole. Using equation (44), it is easy to show that

$$\delta_{ij}(\vec{x})\Psi[\Sigma] = \exp(\frac{ieg}{4\pi}\Omega(\Sigma))\left(\Delta_{ij}(\vec{x}) + ieb_{ij}(\vec{x})\right)\Psi[\gamma]. \tag{47}$$

Then, the Schrödinger equation of the theory can be written down as

$$i\partial_t\,\Psi[\Sigma, t] = \int d\vec{x}^3\left[\frac{1}{2}e^2\left(T^i(\vec{x}, \gamma)\right)^2 - \frac{1}{4e^2}\left(\delta_{ij}(\vec{x})\right)^2\right]\Psi[\Sigma, t], \tag{48}$$

which looks like the the Schrödinger equation of the free (i.e., without charges or monopoles) Maxwell theory in the LR [4,5], except by the presence of the surface derivative instead of the loop derivative, and the surface dependence (instead of simple loop-dependence) of the wave functional. But since the only property of Σ that matters is the solid angle subtended from the monopole, this surface-dependence is topological: if we replace Σ by another surface Σ' that has the same boundary γ, the wave functional changes as

$$\Psi[\Sigma'] = \exp(i\,egp)\Psi[\Sigma], \tag{49}$$

where p is the number of times that the closed surface $S = \Sigma' \circ (-\Sigma)$, that results from the composition of Σ' and the surface opposite to Σ, wraps around the monopole. Thus, we can take the Schrödinger equation of the Maxwell theory with external monopole as that corresponding to the free theory, provided that simultaneously we allow for non-trivial boundary conditions for the loop-dependent wave functionals: every time that the loop goes around a "closed trajectory" that encloses p times the monopole, the wave function picks up the phase factor $\exp(i\,egp)$. The loop-dependent wave functional becomes multivalued due to the presence of the magnetic monopole, and can be finally written down as a free loop-equation

$$i\partial_t\,\Psi[\gamma, t] = \int d\vec{x}^{\,3}\left[\frac{1}{2}e^2\left(T^i(\vec{x}, \gamma)\right)^2 - \frac{1}{4e^2}\left(\Delta_{ij}(\vec{x})\right)^2\right]\Psi[\gamma, t], \tag{50}$$

where the multivalued wave functional $\Psi[\gamma, t]$ obeys the boundary condition

$$\Psi[[S].\gamma] = \exp(i\,egp)\Psi[\gamma]. \tag{51}$$

In this equation, $[S].\gamma$ means that the loop γ has described a "closed trajectory" sweeping the surface S and wrapping p times to the monopole.

This can be understood as a generalization of what happens in ordinary quantum mechanics in multiply connected configuration spaces [7–9]. In such cases, multivaluedness of the wave function is allowed, being restricted to multiplication of the wave function by a phase factor carrying a representation of the fundamental group of the configuration space. But this is precisely what we have found in our study: since the configuration space of our quantum formulation is the space of loops in $R^3 - \{origin\}$, a "point" in the set is a loop, while the "closed curves" swept by loops will be closed surfaces, whose properties of contractibility in $R^3 - \{origin\}$ will define the fundamental group of the configuration space. Now, the phase factor $\exp(i\,egp)$ appearing in equation (49) just corresponds to a one-dimensional representation of this fundamental group, since it classifies the surfaces according to the manner they wrap the monopole.

To conclude, we should mention a feature that could look somewhat striking at first sight. If Dirac's quantization condition (1) holds, the topological phase factor appearing in (49) becomes the unity and the dependence on the surface vanishes. But then the wave functional becomes single-valued and the effect of the monopole disappears at all!. What happens is that, in the absence of electrically charged particles, there is no need to quantize electric or magnetic charges. Recall that in the formulation of Dirac, charge quantization arises when the wave function of the *charged particle* is asked to be single-valued in presence of the monopole. Or, alternatively, when the action functional is asked to be independent of the string attached to the monopole [in our case, this corresponds to demanding that changing f^μ (given in (13)) by any other vector field obeying equation (4) does not modify the action]. Yet, there is another approach, which does not employs vector potentials, that derives charge quantization from the consistency of the Heisenberg equations of motion of a charged particle in the field of a magnetic monopole [18]. But in the absence of electric charges, all this is automatically guaranteed, and Dirac's quantization condition is not required to have a consistent theory.

It would be interesting, in view of the above discussion, to study the LR formulation of the theory in the case with both charges and monopoles. Also, the present approach could be generalized to the study of higher-rank Abelian theories, with their corresponding extended objects generalizing electric and magnetic charges [17].

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